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ON A SUBCLASS OF QUASI-STARLIKE FUNCTIONS

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ABSTRACT. In this paper, we define a new subclass of univalent functions namely QS*(λ) and obtain sharp coefficient bounds, logarithmic coefficients, coefficient of inverse functions and solve Hankel determinant problem for normalized analytic functions.

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1. INTRODUCTION

Let A denote the class of analytic functions f in the unit disc$\mathbb{D}$ ={z ∈ C : |z| <1},with taylor series expansion of the form

$$f(z)= z+ \sum_{n=2}^{\infty} a_n z_n \qquad (1.1)$$

normalised by the conditions $f(0) = f'(0) - 1$.Let S be the subclass of A, consisting of univalent functions.A function $f \in A$ is called starlike, if $f(D)$ is starlike and convex if $f(D)$ is convex.Let S* and C denote the classes of starlike and convex functions in S, respectively.The analytic characterization of S* and C are given by:$f \in$ C iff $Re\frac{zf'(z)}{f'(z)} > 0$ and $f \in$ S* iff $Re\frac{zf'(z)}{f(z)} > 0$.

A function f ∈ A is said to be close-to-star, if and only if there exists a function g(z) ∈ S*

$$\mathrm{Re}\frac{f(z)}{g(z)} > 0$$

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Such a class of functions are denoted by CS*.A function f∈A is said to be in close to convex, if and only if there exists a function g ∈C such that, for z ∈D

$$\mathrm{Re}\frac{f'(z)}{g'(z)}>0$$

The class of close to convex functions are denoted by K. A function f is said to be quasi-convex in D if and only if there exists a convex function g with g(0)=0, g'(0)=1, such that, for z ∈ D

$$\mathrm{Re}\frac{(zf'(z))'}{g'(z)}>0$$

Denote the class of quasi-convex functions by C*. Several authors considered convex combination of the quantities and $\frac{zf'(z)}{f(z)}$ and $\frac{(zf'(z))'}{f'(z)}$ and studied their properties. In [9],Noor and others introduced subclasses of univalent functions of the form

$$\alpha\left(\frac{f'(z)}{g'(z)}\right)+(1-\alpha)\frac{(zf'(z))'}{g'(z)}$$

whose real part is positive with respect to a convex function g , proved very interesting results and pointed out its relation with earlier known classes.The class of α-quasi convex functions is denoted by Q_α. In the year 2018, D.K.Thomas considered a subclass of univalent functions f(z) satisfying

$$Re\left(\frac{zf'(z)}{f(z)}\right)^{1-\lambda}\left(1+\frac{zf'(z)}{f'(z)}\right)^{\lambda} > 0$$

and derived many interesting results. Motivated by the works of earlier authors, we in this paper,introduce a subclass of univalent functions f(z) of the form (1.1) defined as follows.

Definition 1.1.A function f(z) ∈ A is said to be in the class QS*(λ) if there is a starlike function g ∈S* such that

$$Re\left\{\left(\frac{f(z)}{g(z)}\right)^{1-\lambda}\left(\frac{(zf'(z))'}{g'(z)}\right)^{\lambda}\right\} > 0,\quad z\in D, \lambda > 0. \qquad (1.2)$$

We observe the following:

If f = g andλ =1, then QS*(λ) reduces to Q*, the class of quasi-starlike functions introduced and studied by K.I.Noor[9].This implies that every member of QS*(λ) is quasi-starlike and hence it is univalent.If λ=0, then QS*(λ) reduces to the class of close-to-star functions[11] studied by PawelZaprawa.

2. PRELIMINARIES

We shall need the following lemmas to prove our results.

Let P denote the Caratheodary function in $\mathbb{D}$, with taylor expansion

$$p(z)=1+\sum_{n=1}^{\infty} p_n z_n \tag{2.1}$$

satisfying Re p(z) > 0 for z$\in\mathbb{D}$.

Lemma 2.1[6]. For complex valued y, with |y|<1, and for some complex valued ζ with $|\zeta|$<1,

$2p_2= p_1^2+$ y $(4-p_1^2)$

$4p_3= p_1^3+2\,(4-p_1^2)\,p_1 y-(4-p_1^2)\,y^2+2(4\text{-}p_1^2)(1-|y|^2)\,\zeta.$

Lemma 2.2.[5] $|p_n|\leq 2$ for $n\geq 1$ and $|p_2-\frac{\mu}{2}p_1^2|\leq \max\{2,2|\mu-1|\}$.

Lemma 2.3.[1] If $0\leq B\leq 1$ and $B(2B-1)\leq D\leq B$ then

$$|p_3-2Bp_1p_2+Dp_1^3|\leq 2.$$

Lemma 2.4.[1] $|p_3-(1+\mu)\,p_1p_2+\mu p_1^3|\leq \max\{2,2|2\mu-1|\}$.

3. THE COEFFICIENTS OF f(z)

In this section, we study coefficient problem for the function class QS*(λ).

Theorem 3.1. Let f $\in$ QS*(λ), for λ >0 and given by(1.1).Then

$$|a_2|\leq \frac{2(2+\lambda)}{1+3\lambda},$$

$$|a_3|\leq \begin{cases} \frac{9+134\lambda+131\lambda^2+30\lambda^3}{(1+3\lambda)^2(1+8\lambda)}, & 0\leq\lambda\leq 1, \\ \frac{9+116\lambda+149\lambda^2+30\lambda^3}{(1+3\lambda)^2(1+8\lambda)}, & \lambda>1, \end{cases}$$

$$|a_4|\leq \begin{cases} \frac{48+1268\lambda+15376\lambda^2+9520\lambda^3+5000\lambda^4+18360\lambda^5+22356\lambda^6+4320\lambda^7}{3(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}, & 0\leq\lambda\leq 1, \\ \frac{48+872\lambda+17536\lambda^2-3044\lambda^3+2840\lambda^4+31320\lambda^5+22356\lambda^6+4320\lambda^7}{3(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}, & \lambda>1. \end{cases}$$

All the inequalities are sharp.

Proof. Let

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (3.1)$$

From (1.1), we have

$$\left(\frac{f(z)}{g(z)}\right)^{1-\lambda}\left(\frac{zf'(z)'}{g'(z)}\right)^{\lambda} = p(z), \qquad \text{where } p \in P.$$

and we have g as a starlike function. Then there exists a function c(z) ∈ P satisfying

$$\frac{zg'(z)}{g(z)} = c(z) = 1+\sum_{n=1}^{\infty} c_n z^n.$$

Using (3.1), after simple computations, we get $b_2 = c_1$,$b_3 = \frac{c_1^2+c_2}{2}$ and b_4=$\frac{c_3}{3} + \frac{c_1^3}{6} + \frac{c_1 c_2}{2}$.

Then equating the coefficients we get,

$$a_2 = \frac{p_1}{1+3\lambda} + \frac{b_2(1+\lambda)}{1+3\lambda},$$

$$a_3 = \frac{p_2}{1+8\lambda} + \frac{b_3(1+2\lambda)}{1+8\lambda} + \frac{p_1^2(9\lambda-9\lambda^2)}{2(1+3\lambda)^2(1+8\lambda)} + \frac{b_2^2(4\lambda+2\lambda^2-6\lambda^3)}{(1+3\lambda)^2(1+8\lambda)} + \frac{p_1 b_2(1+16\lambda+15\lambda^2)}{(1+3\lambda)^2(1+8\lambda)},$$

$$a_4 = \frac{p_3}{1+15\lambda} + \frac{b_4(1+3\lambda)}{1+15\lambda} + \frac{b_2 b_3(22\lambda+28\lambda^2-50\lambda^3)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)} + \frac{p_1 p_2(24\lambda-24\lambda^2)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)} + \frac{p_2 b_2(1+36\lambda+35\lambda^2)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)} +$$
$$\frac{b_2^3(-76\lambda-263\lambda^2-1694\lambda^3-2368\lambda^4+1026\lambda^5+2349\lambda^6+1080\lambda^7)}{6(1+3\lambda)^3(1+8\lambda)(1+15\lambda)} + \frac{p_1^3(-81\lambda-27\lambda^2-1404\lambda^3+1512\lambda^4)}{6(1+3\lambda)^3(1+8\lambda)(1+15\lambda)} +$$
$$\frac{p_1^2 b_2(-33\lambda+180\lambda^2-1047\lambda^3-180\lambda^4+1080\lambda^5)}{2(1+3\lambda)^3(1+8\lambda)(1+15\lambda)} + \frac{p_1 b_2^2(-34\lambda+107\lambda^2-1045\lambda^3-1323\lambda^4+1215\lambda^5+1080\lambda^6)}{2(1+3\lambda)^3(1+8\lambda)(1+15\lambda)} +$$
$$\frac{p_1 b_3(1+37\lambda+70\lambda^2)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)}.$$

The first inequality for $|a_2|$ is trivial. For $|a_3|$, we have

$$a_3 = \frac{p_2}{1+8\lambda} + \frac{p_1^2(9\lambda-9\lambda^2)}{2(1+3\lambda)^2(1+8\lambda)} + \frac{p_1 c_1(1+16\lambda+15\lambda^2)}{(1+3\lambda)^2(1+8\lambda)} + \frac{c_1^2(1+16\lambda+25\lambda^2+12\lambda^3)}{2(1+3\lambda)^2(1+8\lambda)} + \frac{c_2(1+2\lambda)}{2(1+8\lambda)}.$$

The first inequality for $|a_3|$ is obvious that on noting that the coefficients of p_1^2 and c_1^2 are positive when 0≤ $\lambda \leq 1$, and applying the inequalities $|p_n|$≤ 2 and $|c_n|$≤ 2 for all n=1,2,…

The second inequality follows by a simple application of lemma(2.2) for p_1 and p_2.

Now consider a_4 such that $a_4 = A + B + C + D$, where

A=$\frac{p_3}{1+15\lambda}+\frac{24\lambda(1-\lambda)p_1p_2}{(1+3\lambda)(1+8\lambda)(1+15\lambda)}+\frac{(-81\lambda-27\lambda^2-1404\lambda^3+1512\lambda^4)p_1^3}{6(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}$,

B=$\frac{(1+3\lambda)c_3}{3(1+15\lambda)}+\frac{(1+36\lambda+85\lambda^2+22\lambda^3)c_1c_2}{2(1+3\lambda)(1+8\lambda)(1+15\lambda)}+\frac{(1+10\lambda+367\lambda^2-206\lambda^3-1567\lambda^4+324\lambda^5+2349\lambda^6+1080\lambda^7)c_1^3}{6(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}$,

C = $c_1\left[\frac{p_1^2(-33\lambda+180\lambda^2-1047\lambda^3-180\lambda^4+1080\lambda^5)}{2(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}+\frac{p_2(1+36\lambda+35\lambda^2)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)}\right]$,

and

D = $p_1\left[\frac{c_1^2(1+9\lambda+408\lambda^2-292\lambda^3-693\lambda^4+1215\lambda^5+1080\lambda^6)}{2(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}+\frac{c_2(1+37\lambda+70\lambda^2)}{2(1+3\lambda)(1+8\lambda)(1+15\lambda)}\right]$.

So that $|a_4|\leq |A|+|B|+|C|+|D|$.

Now for A, on the interval $0\leq\lambda\leq 1$, we use lemma (2.1), to express the coefficient of p_2 and p_3 interms of p_1, we get the inequality

|A|≤

$$\frac{p^3(1+11\lambda+321\lambda^2-549\lambda^3+792\lambda^4)}{4(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}+\frac{(4-p^2)p|y|(1+35\lambda)}{2(1+3\lambda)(1+8\lambda)(1+15\lambda)}+\frac{p(4-p^2)|y|^2}{4(1+15\lambda)}+\frac{(4-p^2)(1-|y|^2)}{2(1+15\lambda)}=\phi(p,y).$$

We now use elementary calculus to find the maximum of the above expression.We consider the end points of $[0,2]\times[0,1]$. First note that, for any value of λ,

$$\phi(0,y)=\frac{4(1-|y|^2)}{2(1+15\lambda)}\leq\frac{2}{(1+15\lambda)},$$

$$\phi(2,y)=\frac{2+22\lambda+642\lambda^2-1098\lambda^3+1584\lambda^4}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)},$$

$$\phi(p,0)=\frac{p^3(1+11\lambda+321\lambda^2-549\lambda^3+792\lambda^4)}{4(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}+\frac{(4-p^2)}{2(1+15\lambda)},$$

$$\phi(p,1)=\frac{p^3(1+11\lambda+321\lambda^2-549\lambda^3+792\lambda^4)}{4(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}+\frac{(4-p)^2p(1+35\lambda)}{2(1+3\lambda)(1+15\lambda)(1+15\lambda)}+\frac{p(4-p)^2}{4(1+15\lambda)}.$$

So that the maximum of this will be

$$|A|\leq\frac{2+22\lambda+642\lambda^2-1098\lambda^3+1584\lambda^4}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)},\ \text{for}\ 0\leq\lambda\leq 1.$$

Now for $\lambda\geq 1$,we now use lemma (2.3). First note that $0\leq$B≤ 1 and D$\geq$ Bso that A is of the form

$$A = \frac{1}{1+15\lambda}\left[p_3 - 2Bp_1p_2 + Bp_1^3 + (D-B)p_1^3\right]$$

with $B=\frac{24\lambda(\lambda-1)}{2(1+3\lambda)(1+8\lambda)}$ and $D=\frac{1512\lambda^4-81\lambda-27\lambda^2-1404\lambda^3}{6(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}$.

We see that $D-B \geq 0$, when $\lambda > 1$

$$A=\frac{1}{1+15\lambda}\left[2+\frac{(-9\lambda+333\lambda^2-1188\lambda^3+864\lambda^4)p_1^3}{6(1+3\lambda)^3(1+8\lambda)}\right].$$

By using the inequality $|p_3 - 2Bp_1p_2 + Dp_1^3| \leq 2$ and again using the inequality $|p_n| \leq 2$, for all n=1,2,… we get the following

$$|A| \leq \frac{2+22\lambda+642\lambda^2-1098\lambda^3+1584\lambda^4}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}.$$

So that for |A|,

$$|A| \leq \begin{cases} \frac{2+22\lambda+642\lambda^2-1098\lambda^3+1584\lambda^4}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}, & 0 \leq \lambda \leq 1 \\ \frac{2+22\lambda+642\lambda^2-1098\lambda^3+1584\lambda^4}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}, & \lambda \geq 1 \end{cases}$$

For B, since the coefficients of c_1c_2 and c_1^3 are positive when $\lambda \geq 0$, using the inequality $|c_n| \leq 2$ for all n=1,2,…we get the result

$$|B| \leq \frac{2(6+166\lambda+1814\lambda^2+2696\lambda^3+502\lambda^4+1890\lambda^5+4698\lambda^6+2160\lambda^7)}{3(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}.$$

On the interval $0 \leq \lambda \leq 1$, we using the lemma(2.2) we have

$$C = c_1\left[\frac{1+36\lambda+35\lambda^2}{(1+3\lambda)(1+8\lambda)(1+15\lambda)}\left(p_2 - p_1^2\,\frac{33\lambda-180\lambda^2+1047\lambda^3+180\lambda^4-1080\lambda^5}{2(1+3\lambda)^2(1+36\lambda+35\lambda^2)}\right)\right].$$

Then we get the result

$$|C| \leq \frac{4(1+36\lambda+35\lambda^2)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)}, \text{ when } 0 \leq \lambda \leq 1.$$

since the coefficient of p_1^2 and p_2 are positive when $\lambda \geq 1$, applying the inequalities of $|p_n| \leq 2$ and $|c_n| \leq 2$ for all n=1,2,…,we get

$$|C| \leq \frac{4+36\lambda+1760\lambda^2-2052\lambda^3+540\lambda^4+4320\lambda^5}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}.$$

So that for C,

$$|C| \leq \begin{cases} \frac{4(1+36\lambda+35\lambda^2)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)} & , \ 0 \leq \lambda \leq 1 \\ \frac{4+36\lambda+1760\lambda^2-2052\lambda^3+540\lambda^4+4320\lambda^5}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)} & , \ \lambda \geq 1 \end{cases}$$

For |D|, since the coefficient of c_1^2 are positive when $\lambda \geq 0$, applying the inequalities $|p_n| \leq 2$ and $|c_n| \leq 2$ for all n=1,2,...We get,

$$|D| \leq \frac{2(3+61\lambda+1117\lambda^2+169\lambda^3-756\lambda^4+2430\lambda^5+2160\lambda^6)}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}.$$

By adding all these on the interval $0 \leq \lambda \leq 1$, we get the first inequality of $|a_4|$. As well as we get the second inequality of $|a_4|$, by adding all these results on the interval $\lambda \geq 1$.

The inequality $|a_2|$ is sharp with respect to p(z)=$\frac{1+z}{1-z}$ and g(z)=$\frac{z}{(1-z)^2}$.

The first inequality of $|a_3|$ is sharp with respect to p(z)=$\frac{1+z}{1-z}$ and g(z)=$\frac{z}{(1-z)^2}$ and the second inequality of $|a_4|$ is sharp with respect to p(z)=$\frac{1+z}{1-z}$ and g(z)=$\frac{z}{(1-z)^2}$.

4. THE COEFFICIENTS OF log f(z)

The logarithmic coefficients $\delta_n$of $f \in S$ are defined by

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} nz^n. \qquad (4.1)$$

These coefficients play an important role for various estimates in the theory of univalent functions. For example, Koebefunction k(z)= z$(1-e^{i\theta}z)^{-2}$for each θ has logarithmic coefficients$\delta_n(k) = e^{\frac{in\theta}{n}}, n > 1$.The inequality $|\delta_n| \leq \frac{1}{n}$ holds f ∈ Sand $|\delta_n| \leq \frac{1}{2n}$ for f ∈ C.

Theorem 4.1. Let f ∈ QS*(λ), g ∈S* for $\lambda \geq 0$ and the coefficients of $\log\left(\frac{f(z)}{z}\right)$ be given by 4.1. Then

$$|\delta_1| \leq \frac{2+\lambda}{1+3\lambda},$$

$$|\delta_2| \leq \begin{cases} \frac{3+60\lambda+83\lambda^2+14\lambda^3}{2(1+3\lambda)^2(1+8\lambda)}, & 0 \leq \lambda \leq 4.6, \\ \frac{3+48\lambda+67\lambda^2+18\lambda^3}{2(1+3\lambda)^2(1+8\lambda)}, & \lambda \geq 4.6, \end{cases}$$

$$|\delta_3| \leq \begin{cases} \frac{4+243\lambda+911\lambda^2+5678\lambda^3+2195\lambda^4-6270\lambda^5-1728\lambda^6+2160\lambda^7}{3(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}, & 0 \leq \lambda \leq 1, \\ \frac{4+429\lambda+935\lambda^2+10917\lambda^3+3735\lambda^4-12750\lambda^5-1782\lambda^6+2160\lambda^7}{3(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}, & \lambda \geq 1. \end{cases}$$

The inequality $|\delta_1|$ is sharp.

Proof. First note that differentiating (4.1), and equating the coefficient gives

$|\delta_1| = \frac{1}{2} a_2$, $|\delta_2| = \frac{1}{2}(a_3 - a_2^2)$, and $|\delta_3| = \frac{1}{2}(a_4 - a_2 a_3 + \frac{1}{3} a_2^2)$.

The inequality of $|\delta_1|$ is trivial.

For $|\delta_2|$, substituting for $a_2$and a_3, we obtain

$$\delta_2 = \frac{1}{2}\left[\frac{p_2}{(1+8\lambda)} + \frac{p_1^2(\lambda-9\lambda^2-1)}{2(1+3\lambda)^2(1+8\lambda)} + \frac{c_2(1+2\lambda)}{2(1+8\lambda)} + \frac{c_1^2(6\lambda+8\lambda^2-2\lambda^3)}{2(1+3\lambda)^2(1+8\lambda)} + \frac{p_1 c_1(7\lambda+7\lambda^2)}{(1+3\lambda)^2(1+8\lambda)}\right].$$

since all the coefficients are positive on $0 \leq \lambda \leq 4$, using inequalities $|p_n| \leq 2$ and $|c_n| \leq 2$ for all n=1,2,…, we get the first inequality of $|\delta_2|$ and for$\lambda \geq 4.6$ using lemma(2.2) with$\mu = \frac{2(2\lambda^3-6\lambda-8\lambda^2)}{(1+3\lambda)^3(1+2\lambda)}$, we get the second inequality of $|\delta_2|$.

For $|\delta_3|$, we again substitute from (3.1) to obtain. Here δ_3=F+G+H+J.

where $F = \frac{p_3}{1+15\lambda} + \frac{p_1 p_2(-1+9\lambda-24\lambda^2)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)} + \frac{p_1^3(2-62\lambda-165\lambda^2-999\lambda^3+1512\lambda^4)}{6(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}$,

$G = \frac{c_3(1+3\lambda)}{3(1+15\lambda)} + \frac{c_1 c_2(9\lambda+19\lambda^2-47\lambda^3)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)} + \frac{c_1^3(-34\lambda-137\lambda^2-1284\lambda^3-2214\lambda^4+294\lambda^5+2349\lambda^6+1080\lambda^7)}{6(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}$,

$H = c_1\left[\frac{p_2(20\lambda+20\lambda^2)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)} + \frac{p_1^2(-56\lambda-179\lambda^2-1248\lambda^3-45\lambda^4+1080\lambda^5)}{2(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}\right]$,

and

$$J = p_1\left[\frac{c_1^2(-36\lambda-95\lambda^2-1107\lambda^3-993\lambda^4-1215\lambda^5+1080\lambda^6)}{2(1+3\lambda)^3(1+8\lambda)(1+15)} + \frac{c_2(10\lambda+20\lambda^2)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)}\right].$$

Therefore $|\delta_3| \leq$ |F|+|G|+|H|+|J|.

For |F|, by applying lemma(2.3) with $B = \frac{1-9\lambda+24\lambda^2}{2(1+3\lambda)(1+8\lambda)}$ and $D = \frac{2-62\lambda-165\lambda^2-999\lambda^3+1512\lambda^4}{6(1+3\lambda)^3(1+8\lambda)}$,

we get

$$|F| \leq \frac{2}{1+15\lambda}, \text{ when } \lambda \geq 0.$$

For |G|,First note that, since the coefficients of c_1^3, c_1c_2 and c_3 are all positive on $1 \leq \lambda \leq 5.184$, using the inequality $|c_n| \leq 2$ for all n =1,2,...

We get the result

$$G = \frac{2+12\lambda+628\lambda^2-1764\lambda^3-5202\lambda^4+2040\lambda^5+9396\lambda^6+4320\lambda^7}{3(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}.$$

Next, write the expression,

$$G = \frac{1+3\lambda}{3(1+15)}\left[c_3 - \frac{3(4\lambda^3-9\lambda-19\lambda^2)c_1c_2}{(1+3\lambda)^2(1+8\lambda)} + \frac{(-34\lambda-137\lambda^2-1248\lambda^3-2214\lambda^4+294\lambda^5+2349\lambda^6+1080\lambda^7)c_1^3}{2(1+3\lambda)^4(1+8\lambda)}\right].$$

It is of the form$\frac{1+3\lambda}{3(1+15\lambda)}[c_3 - 2Bc_1c_2 + Dc_1^3]$.

On the interval $\lambda \geq 5.184$, we note that $0 \leq B \leq 1$ and $(D - B) \geq 0$.

where$B=\frac{3(4\lambda^3-9\lambda-19\lambda^2)}{2(1+3\lambda)^2(1+8\lambda)}$ and $D=\frac{(-34\lambda-137\lambda^2-1248\lambda^3-2214\lambda^4+294\lambda^5+2349\lambda^6+1080\lambda^7)}{2(1+3\lambda)^4(1+8\lambda)}$.

Applying lemma(2.3), we get the result

$$G = \frac{2+12\lambda+628\lambda^2-1764\lambda^3-5202\lambda^4+2040\lambda^5+9396\lambda^6+4320\lambda^7}{3(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}.$$

For the remaining interval $0 \leq \lambda \leq 1$, we use lemma (2.1)

$$|G| \leq \frac{c^3(1+6\lambda+314\lambda^2-882\lambda^3-2601\lambda^4+1020\lambda^5+4698\lambda^6+2160\lambda^7)}{12(1+3\lambda)^3(1+8\lambda)(1+15\lambda)} + \frac{c|y|(4-c^2)(2+82\lambda+228\lambda^2+120\lambda^3)}{12(1+3\lambda)(1+8\lambda)(1+15\lambda)} + \frac{c|y|^2(4-c^2)(1+3\lambda)}{12(1+15\lambda)} + \frac{(4-c^2)(1-|y|^2)(1+3\lambda)}{6(1+15\lambda)} = \phi(c,|y|).$$

We now use elementary calculus to find the maximum of the above expression.

We consider the end points [0,2]×[0,1]

$$\phi(0,|y|) = \frac{4(1-|y|^2)(1+3\lambda)}{6(1+15\lambda)} \leq \frac{2(1+3\lambda)}{3(1+3\lambda)},$$

$$\phi(2,|y|) = \frac{2(1+6\lambda+314\lambda^2-882\lambda^3-2601\lambda^4+1020\lambda^5+4698\lambda^6+2160\lambda^7)}{3(1+3\lambda)^3(1+8\lambda)(1+15\lambda)},$$

$$\phi(c,0) = \frac{c^3(1+6\lambda+314\lambda^2-882\lambda^3-2601\lambda^4+1020\lambda^5+4698\lambda^6+2160\lambda^7)}{12(1+3\lambda)^3(1+8\lambda)(1+15\lambda)} + \frac{c(4-c^2)(2+82\lambda+228\lambda^2+120\lambda^3)}{12(1+3\lambda)(1+8\lambda)(1+15\lambda)},$$

$$\phi(c,1)=\frac{c^3(1+6\lambda+314\lambda^2-882\lambda^3-2601\lambda^4+1020\lambda^5+4698\lambda^6+2160\lambda^7)}{12(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}+\frac{c(4-c^2)(2+82\lambda+228\lambda^2+120\lambda^3)}{12(1+3\lambda)(1+8\lambda)(1+15\lambda)}+\frac{c(4-c^2)(1+3\lambda)}{12(1+15\lambda)}.$$

So that for $0\leq \lambda \leq 1$,

$$G=\frac{2+12\lambda+628\lambda^2-1764\lambda^3-5202\lambda^4+2040\lambda^5+9396\lambda^6+4320\lambda^7}{3(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}.$$

For $\lambda \geq 0$,

$$G=\frac{2+12\lambda+628\lambda^2-1764\lambda^3-5202\lambda^4+2040\lambda^5+9396\lambda^6+4320\lambda^7}{3(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}.$$

For $|H|$, by applying lemma(2.2), with $\mu=\frac{56\lambda+179\lambda^2+1248\lambda^3+45\lambda^4-1080\lambda^5}{(1+3\lambda)^2}$.

We get the result,

$$|H|\leq\frac{4(20\lambda+20\lambda^2)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)},\qquad \text{for } 0\leq\lambda\leq 1.176.$$

since the coefficients of p_1^2, p_2 and c_2 all are positive on $\lambda \geq 1.176$, applying the inequality $|p_n|\leq 2$ and $|c_n|\leq 2$ for all n=1,2,..., we get the result

$$|H|\leq\frac{144\lambda+156\lambda^2+3792\lambda^3+540\lambda^4-4320\lambda^5}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)},\qquad \text{for } \lambda\geq 1.176.$$

For $|J|$, On the interval $0\leq \lambda \leq 1$, using lemma(2.2), we get

$$|J|\leq\frac{20\lambda+40\lambda^2}{(1+3\lambda)(1+8\lambda)(1+15\lambda)}.$$

For $\lambda \geq 1$, since the coefficients of c_1^2, c_2 and p_2 all are positive , we are using the inequalities $|p_n|\leq 2$ and $|c_n|\leq 2$ for all n=1,2,...

We get the result,

$$|J|\leq\frac{104\lambda+60\lambda^2+3588\lambda^3+3252\lambda^4+4860\lambda^5+4320\lambda^6}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)},\ \text{for } \lambda\geq 1.$$

By adding all these, we get the first and second inequalities of $|\delta_3|$ on the intervals $0\leq \lambda \leq 1$ and $\lambda \geq 1$ respectively.

The inequality $|\delta_1|$ is sharp with respect to $p(z)=\frac{1+z}{1-z}$ and $g(z)=\frac{z}{(1-z)^2}$.

5. THE COEFFICIENTS OF INVERSE FUNCTION

As f ∈ QS*(λ) is univalent, its inverse function f^{-1} exist defined in some disc $|\omega| \leq \lambda$. Let $f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \ldots$ Then since $f(f^{-1}(\omega)) = \omega$, equating the coefficient gives

$$A_2 = -a_2,$$

$$A_3 = 2a_2^2 - a_3,$$

$$A_4 = -5a_2^3 + 5a_2a_3 - a_4.$$

Theorem 5.1.Let f ∈ QS*(λ), g ∈ S*, for $\lambda \geq 0$, and f^{-1} be the inverse function of f.

Then

$$|A_2| \leq \frac{2(2+\lambda)}{(1+3\lambda)},$$

$$|A_3| \leq \begin{cases} \frac{24+66\lambda+174\lambda^2+52\lambda^3}{(1+3\lambda)^2(1+8\lambda)}, & 0 \leq \lambda \leq 2, \\ \frac{16+108\lambda+128\lambda^2+36\lambda^3}{(1+3\lambda)^2(1+8\lambda)}, & \lambda \geq 2, \end{cases}$$

$$|A_4| \leq \frac{540+9638\lambda+37798\lambda^2+26482\lambda^3+24590\lambda^4+32220\lambda^5+9396\lambda^6+4320\lambda^7}{3(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}, \lambda \geq 0.$$

The first inequality is sharp.

Proof.The inequality $|A_2|$ is trivial.

For $|A_3|$ write,

$$|A_3| = \frac{p_1^2(4+23\lambda+9\lambda^2)}{2(1+3\lambda)^2(1+8\lambda)} - \frac{p_2}{(1+8\lambda)} + \frac{b_2^2(1+8\lambda+16\lambda^2+11\lambda^3)}{(1+3\lambda)^2(1+8\lambda)} - \frac{b_3(1+2\lambda)}{1+8\lambda} + \frac{p_1b_2(3+20\lambda+17\lambda^2)}{(1+3\lambda)^2(1+8\lambda)}.$$

Applying lemma(2.2), on the interval $0 \leq \lambda \leq 2$. We get

$$|A_3| \leq \frac{24+66\lambda+174\lambda^2+52\lambda^3}{(1+3\lambda)^2(1+8\lambda)}.$$

Since p_1^2, p_2, c_1^2, c_2 are positive,by using lemma(2.2),we get

$$|A_3| \leq \frac{16+108\lambda+128\lambda^2+36\lambda^3}{(1+3\lambda)^2(1+8\lambda)}, \text{ when } \lambda \geq 2.$$

Consider A_4 =M+N+P+Q.

$$M=\frac{p_3}{1+15\lambda}-\frac{p_1p_2(5+51\lambda+24\lambda^2)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)}+\frac{p_1^3(30+474\lambda+1683\lambda^2+621\lambda^3+1512\lambda^4)}{6(1+3\lambda)^3(1+8\lambda)(1+15\lambda)},$$

$$N=\frac{c_3(1+3\lambda)}{3(1+15\lambda)}-\frac{c_1c_2(4+54\lambda+150\lambda^2+128\lambda^3)}{2(1+3\lambda)^2(1+8\lambda)}+$$
$$\frac{c_1^3(16+310\lambda+1687\lambda^2+3004\lambda^3+2858\lambda^4+2574\lambda^5+2349\lambda^6+1080\lambda^7)}{2(1+3\lambda)^4(1+8\lambda)}$$

$$P=c_1\left[\frac{4+44\lambda+40\lambda^2}{(1+3\lambda)(1+8\lambda)(1+15\lambda)}\left(p_2-\frac{p_1^2(20+332\lambda+1245\lambda^2+348\lambda^3+495\lambda^4+1080\lambda^5)}{2(1+3\lambda)^2(4+44\lambda+40\lambda^2)}\right)\right],$$

and $Q=p_1\left[\frac{c_1^2(20+376\lambda+1797\lambda^2+2225\lambda^3+1287\lambda^4+1215\lambda^5+1080\lambda^6)}{2(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}+\frac{c_2(10\lambda+20\lambda^2)}{(1+3\lambda)(1+8\lambda)(1+15\lambda)}\right].$

Hence $|A_4|$≤|M|+|N|+|P|+|Q|.

For |M|, since the coefficients of p_1^3, p_1p_2 and p_3 are positive when $\lambda \geq 0$,by applying lemma(2.2) we get the result

$$|M| \leq \frac{22+342\lambda+142\lambda^2-1098\lambda^3+1584\lambda^4}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}.$$

For |N|,we use lemma(2.3) when $\lambda \geq 0$ with $B=\frac{3(4+54\lambda+150\lambda^2+128\lambda^3)}{4(1+3\lambda)^2(1+8\lambda)}$ and $D=\frac{16+310\lambda+1687\lambda^2+3004\lambda^3+2858\lambda^4+2574\lambda^5+2349\lambda^6+1080\lambda^7}{2(1+3\lambda)^4(1+8\lambda)}$.We get

$$|N| \leq \frac{42+812\lambda+3988\lambda^2+4012\lambda^3+614\lambda^4+4680\lambda^5+9396\lambda^6+4320\lambda^7}{3(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}.$$

For |P|, using lemma(2.2), with $\mu=\frac{20+332\lambda+1245\lambda^2+348\lambda^3+495\lambda^4+1080\lambda^5}{(1+3\lambda)^2(4+44\lambda+40\lambda^2)}$

we get the result, by using lemma(2.2)

$$|P|\leq \frac{64+1056\lambda+3620\lambda^2-1152\lambda^3+540\lambda^4+4320\lambda^5}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}, \text{ when } \lambda \geq 0.$$

For |Q|, on the interval $\lambda \geq 0$ since all the coefficients are positive on the interval we applying lemma(2.2), we get the result

$$|Q| \leq \frac{2(40+772\lambda+3754\lambda^2+4870\lambda^3+2934\lambda^4+2430\lambda^5+2160\lambda^6)}{(1+3\lambda)^3(1+8\lambda)(1+15\lambda)}.$$

Therefore adding up all these results, we get the inequality for $|A_4|$.

The inequality $|A_2|$ is sharp with respect to p(z)=$\frac{1+z}{1-z}$ and g(z)= $\frac{z}{(1-z)^2}$.

6. THE SECOND HANKEL DETERMINANT

The problem of finding sharp bounds for the Hankel determinant $H_2(2) = |a_2 a_4 - a_3^2|$ for subclass of univalent functions has received much attention in recent years.

Theorem 6.1. Let f ∈QS*(λ) for $0 \leq \lambda \leq 1$, and be given by (1.1)

Then

$$H_2(2) = \begin{cases} \frac{163+4843\lambda+30865\lambda^2+68661\lambda^3+52440\lambda^4+1620\lambda^5}{12(1+3\lambda)^2(1+8\lambda)^2(1+15\lambda)}, & \lambda \neq 1, \\ \frac{1}{8} & , \lambda = 1. \end{cases} \qquad (6.1)$$

Here $H_2(2) = |a_2 a_4 - a_3^2|$. On substituting a_2, a_3 and a_4 values in above expression ,we get

$H_2(2)$=V+W.Where

$$V= \frac{p_4(13\lambda+133\lambda^2+678\lambda^3+1800\lambda^4-3168\lambda^5)}{2(1+3\lambda)^4(1+8\lambda)^2(1+15\lambda)} + \frac{(13\lambda+55\lambda^2)p^2(4-p^2)|y|}{2(1+3\lambda)^2(1+8\lambda)(1+15\lambda)} + \frac{p^2(4-p^2)|y|^2}{4(1+3\lambda)(1+15\lambda)} + \frac{(4-p^2)|y|^2}{4(1+8\lambda)^2} +$$

$$\frac{p(4-p^2)(1-|y|^2)}{2(1+3\lambda)(1+15\lambda)} + \frac{(4-c^2)^2(1-|y|^2)(1+2\lambda)^2}{16(1+8\lambda)^2} + \frac{c(4-c^2)(1-|y|^2)(1+\lambda)}{6(1+15\lambda)} +$$

$$\frac{c^4(-3-245\lambda-3454\lambda^2-27350\lambda^3+128911\lambda^4-235721\lambda^5-91504\lambda^6+197748\lambda^7+228096\lambda^8+69120\lambda^9)}{48(1+3\lambda)^4(1+8\lambda)^2(1+15\lambda)} +$$

$$\frac{c^2(4-c^2)|y|(1+89\lambda+743\lambda^2+1927\lambda^3+1764\lambda^4+660\lambda^5)}{24(1+3\lambda)^2(1+8\lambda)^2(1+15\lambda)} + \frac{c^2|y|^2(4-c^2)(1+\lambda)}{12(1+15\lambda)},$$

and

$$W= p_1\left[\frac{c_3(1+3\lambda)}{3(1+3\lambda)(1+15\lambda)} + \frac{c_1c_2(24\lambda+150\lambda^2+158\lambda^3-164\lambda^4)}{2(1+3\lambda)^2(1+8\lambda)^2(1+15\lambda)} + \frac{c_1^3(-2-210\lambda-2718\lambda^2-17430\lambda^3-64352\lambda^4)}{6(1+3\lambda)^4(1+8\lambda)^2(1+15\lambda)} + \right.$$

$$\left.\frac{c_1^3(-77216\lambda^5+16254\lambda^6+78192\lambda^7+34560\lambda^8)}{6(1+3\lambda)^4(1+8\lambda)^2(1+15\lambda)}\right] + c_2\left[\frac{p_1^2(1+36\lambda+222\lambda^2+443\lambda^3+270\lambda^4)}{2(1+3\lambda)^2(1+8\lambda)^2(1+15\lambda)} - \frac{p_2(1+2\lambda)}{(1+8\lambda)^2}\right] +$$

$$c_1^2\left[\frac{p_1^2(-1-119\lambda-1439\lambda^2-8365\lambda^3-27086\lambda^4-17376\lambda^5+19890\lambda^6+17280\lambda^7)}{2(1+3\lambda)^4(1+8\lambda)^2(1+15\lambda)} + \right.$$

$$\left.\frac{p_2(14\lambda+102\lambda^2+222\lambda^3+190\lambda^4)}{(1+3\lambda)^2(1+8\lambda)^2(1+15\lambda)}\right] +$$

$$c_1\left[\frac{p_3(1+\lambda)}{(1+3\lambda)(1+15\lambda)} + \frac{p_1p_2(-1+6\lambda+5\lambda^2-194\lambda^3-192\lambda^4)}{(1+3\lambda)^2(1+8\lambda)^2(1+15\lambda)} + \right.$$

$$\left.\frac{p_1^3(-234\lambda-2628\lambda^2-13212\lambda^3-35388\lambda^4+13446\lambda^5+38016\lambda^6)}{6(1+3\lambda)^2(1+8\lambda)^2(1+15\lambda)}\right].$$

We simplify V and W by converting p_2 and p_3 interms of p_1 using lemma (2.1) and then normalize p_1= p over $0 < p < 2$. Further proceeding by using the lemma (2.2), we arrived at the required inequality for $\lambda \neq 1$.When λ=1, QS* =C*. Clearly every quasi-convex function is close-to-convex[7] and $H_2(2)$ for K is$\frac{1}{8}$, which is sharp[8].

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